Minimum Detection Efficiencies for a Loophole–Free Bell–type Test

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We discuss the problem of finding the most favorable conditions for closing the detection loophole in a test of local realism with a Bell inequality. For a generic non–maximally entangled two–qubit state and two alternative measurement bases we apply Hardy's proof of non–locality without inequality and derive an Eberhard–like inequality. For an infinity of non–maximally entangled states we find that it is possible to refute local realism by requiring perfect detection efficiency for only one of the two measurements: the test is free from the detection loophole for any value of the detection efficiency corresponding to the other measurement. The maximum tolerable noise in a loophole–free test is also evaluated.

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According to Bell theorem [1], the quantum–mechanical correlations shown in an ideal experiment by the separate parties of an entangled state are so strong that cannot be reproduced by any local realistic model.

Several experiments with Bell inequalities [2–4] have been performed to test local realism vs quantum mechanics [5–9]. No one of these experiments allowed a conclusive refutation of local realism, i.e., the violation of a genuine Bell inequality. Local hidden–variable models exploiting the non–ideal behaviour of the apparata exist which reproduce the results obtained in each one of these tests [3,10].

The reason behind the impossibility of a conclusive Bell–type test is the persistence of the locality and the detection loopholes. For a bipartite system, the locality loophole arises when the two (left and right) joint measurements required in the Bell–test are performed in space–time regions which are not space–like separated from each other. In this case, one cannot exclude that information on the measurement settings is exchanged between the two measurement regions. Besides, when the detection efficiency in a Bell–test is smaller than a certain critical value, or the test suffers from noise beyond a certain threshold, a local realistic model can be constructed so as to reproduce the quantum–mechanical correlations entering the Bell inequality. In this case, the test is affected by the detection loophole, and only non–genuine Bell inequalities (incorporating supplementary assumptions) can be tested experimentally.

For any bipartite and entangled state one can derive Bell inequalities without the introduction of (plausible but not testable) supplementary assumptions concerning undetected events [3,4,11]. In particular, the most appropriate inequality for confronting local realism vs quantum mechanics was derived long ago by Clauser and Horne [3]. For maximally (non–maximally) entangled states, if one assumes that all the involved measurements are performed with the same overall detection efficiency η, these Clauser–Horne inequalities are violated by quantum mechanics only if $\eta > 2/(1+\sqrt{2}) \simeq 0.83$ [12] ($\eta > 2/3 \simeq 0.67$ [13]).

Only the recent tests with entangled ions of Refs. [8,9] closed the detection loophole. On the contrary, the locality loophole was closed using entangled photons [6]. No experiment closing simultaneously both the detection and the locality loopholes has been performed so far.

To find a solution to the detection loophole problem, two approaches are possible: one can either identify apparata and detectors allowing the highest detection efficiencies (see for instance the use of homodyne detection in continuous–variables Bell–tests [14]) or search for new Bell inequalities and/or entangled states allowing the use of less efficient detectors and sustaining the maximum amount of noise, as recently done in Ref. [15–19]. Here we further consider the question of performing a Bell–test with inefficient experimental apparata. By using bipartite non–maximally entangled states for which Hardy's proof of Bell theorem without inequalities applies, we demonstrate that it is possible to perform a genuine Bell–type test by requiring perfect detection efficiency for only one of the two observables to be alternatively measured on each one of the two parties.

We emphasize that, apart form the obvious importance for the foundations of quantum mechanics, the question of a genuine violation of Bell inequalities is also relevant in connection with quantum information theory. Indeed, the existence of some secure quantum key distribution protocols is closely related to the loophole–free violation of Bell inequalities [20].

We start our discussion by determining the most general non–maximally entangled state suitable for proving Hardy's contradiction without inequalities between local realism and quantum mechanics [21]. Let us introduce two incompatible qubit bases $\{|a_+\rangle, |a_-\rangle\}$ and $\{|b_+\rangle, |b_-\rangle\}$ of eigenvectors of the observables $\hat{a}_\pm = |\pm_a\rangle\langle\pm_a|$ and $\hat{b}_\pm = |\pm_b\rangle\langle\pm_b|$ with eigenvalues $a_\pm = \pm 1$ and $b_\pm = \pm 1$. In general, the two bases are related to each other by:

$$
\begin{aligned}
|+_b\rangle &= \alpha|+_a\rangle + \beta e^{i\phi}|-_a\rangle \ , \\
|-_b\rangle &= -\beta e^{-i\phi}|+_a\rangle + \alpha|-_a\rangle \ ,
\end{aligned} \tag{1}
$$

α and β being real numbers with $\alpha^2+\beta^2=1$.

Hardy's proof [21] is applied to a set of four joint probabilities, three of which are vanishing; we choose to use the following quantum–mechanical values:

$$P^{\rm I}_{\rm QM}(a_+,a_+) \neq 0 \ , \tag{2}$$
$$P^{\rm I}_{\rm QM}(a_+,b_-) = 0 \ , \tag{3}$$
$$P^{\rm I}_{\rm QM}(b_-,a_+) = 0 \ , \tag{4}$$
$$P^{\rm I}_{\rm QM}(b_+,b_+) = 0 \ , \tag{5}$$

where the index I reminds us that we are considering the ideal case of perfect experimental apparata.

The most general non–maximally entangled state satisfying the prediction of Eq. (5) is:

$$|\psi\rangle = A|+_b\rangle|-_b\rangle + B|-_b\rangle|+_b\rangle + C|-_b\rangle|-_b\rangle \ , \tag{6}$$

where $|A|^2+|B|^2+|C|^2=1$.

The vanishing values of the joint probabilities of Eqs. (3) and (4) require $A\alpha - C\beta e^{-i\phi}=0$ and $B\alpha - C\beta e^{-i\phi}=0$, respectively. The solution of these equations which satisfies the normalization to one of $|\psi\rangle$ and of the two qubit bases $\{|a_+\rangle,|a_-\rangle\}$ and $\{|b_+\rangle,|b_-\rangle\}$ is:

$$A = B = \sqrt{\frac{1-\alpha^2}{2-\alpha^2}}e^{-i\phi} \ , \tag{7}$$
$$C = \frac{\alpha}{\sqrt{2-\alpha^2}} \ . \tag{8}$$

The state $|\psi\rangle$ for which the three quantum–mechanical predictions (3)–(5) are fulfilled, which is called Hardy's state, is thus:

$$
\begin{aligned}
|\psi_H\rangle = \frac{1}{\sqrt{2-\alpha^2}} \Big[& \sqrt{1-\alpha^2}e^{-i\phi}(|+_b\rangle|-_b\rangle + |-_b\rangle|+_b\rangle) \\
& +\alpha|-_b\rangle|-_b\rangle \Big] \ ,
\end{aligned} \tag{9}
$$

while Hardy's fraction (i.e., the non–vanishing probability of Hardy's reasoning) turns out to be:

$$P^{\rm I}_{\rm QM}(a_+,a_+) = \frac{(1-\alpha^2)^2\alpha^2}{2-\alpha^2} \ , \tag{10}$$

and assumes the maximum value of $(5\sqrt{5}-11)/2 \simeq 0.0902$ when $\alpha=\alpha_H \equiv \sqrt{(3-\sqrt{5})/2} \simeq 0.618$.

The contradiction without inequalities between local realism and quantum mechanics applies to the case of ideal measurements and consist in showing that there is no local realistic theory which reproduces the predictions of Eqs. (3), (4), (5) and (10). This has been proved in Ref. [21].

Moreover, a Bell inequality in the form due to Eberhard [13] can be deduced which generalizes Hardy's incompatibility proof to the case of a real test [22], which has to confirm null events with imprecise state preparation and measurements. Three different Eberhard inequalities correspond to the incompatibility proof adopting the joint probabilities (3), (4), (5) and (10). The more convenient inequality for closing the detection loophole turns out to be:

$$
\begin{aligned}
H_{\rm LR} \equiv P(a_+,a_+)/ & \left[P(a_+,b_-)+P(b_+,b_+)\right. \\
& \left.+P(b_-,a_+)+P(a_+,b_0)+P(b_0,a_+)\right] \leq 1 \ ,
\end{aligned} \tag{11}
$$

where the outcome denoted by b_0 corresponds to the cases in which, due to imperfect experimental apparata, the measurement in the $\{|b_+\rangle,|b_-\rangle\}$ basis does not produce an outcome.

The previous Eberhard inequality can be equivalently rewritten in the form of a Clauser–Horne inequality [3]:

$$
\begin{aligned}
& P(a_+,a_+)+P(a_+,b_+)+P(b_+,a_+)-P(b_+,b_+) \\
& \leq P(a_+,*)+P(*,a_+) \ ,
\end{aligned} \tag{12}
$$

with the single–side probabilities given by:

$$
\begin{aligned}
P(a_+,*) &= P(a_+,b_+)+P(a_+,b_-)+P(a_+,b_0) \ , \\
P(*,a_+) &= P(b_+,a_+)+P(b_-,a_+)+P(b_0,a_+) \ .
\end{aligned} \tag{13}
$$

The quantum–mechanical values of the non–vanishing probabilities appearing in Eberhard inequality are:

$$P_{\rm QM}(a_+,a_+) = \frac{(1-\alpha^2)^2\alpha^2}{2-\alpha^2}\eta_{L,a}\,\eta_{R,a} \ , \tag{14}$$
$$P_{\rm QM}(a_+,b_0) = \frac{(1-\alpha^2)^2}{2-\alpha^2}\eta_{L,a}(1-\eta_{R,b}) \ , \tag{15}$$
$$P_{\rm QM}(b_0,a_+) = \frac{(1-\alpha^2)^2}{2-\alpha^2}\eta_{R,a}(1-\eta_{L,b}) \ , \tag{16}$$

where we have considered different overall detection efficiencies for the four measurements (two on the left (L) and two on the right (R)) involved in the inequality.

Inequality (11) (and (12)) is thus violated by quantum mechanics when the four detection efficiencies of the problem satisfy:

$$H_{\rm QM} = \frac{\alpha^2\eta_{L,a}\eta_{R,a}}{\eta_{L,a}(1-\eta_{R,b})+\eta_{R,a}(1-\eta_{L,b})} > 1 \ . \tag{17}$$

Let us considered the following special cases:
Case 1: $\eta \equiv \eta_{L,a}=\eta_{R,a}=\eta_{L,b}=\eta_{R,b}$, as for photon–photon [5–7] and atom–atom [23] entanglement. Eq. (17) is satisfied when

$$\eta > 2/(2+\alpha^2) \ . \tag{18}$$

The minimum value of the detection efficiency, $\eta_{\rm min} = 2/3 \simeq 0.67$, is found for $\alpha=1$. Note however that α cannot be identically equal to 1, otherwise our entangled state would be a factorized one: $|\psi_H\rangle \to |-_b\rangle|-_b\rangle = |-_a\rangle|-_a\rangle$, and all probabilities entering Eberhard inequality would be vanishing. Note also that the above

result for $\eta_{\min}$ is analogous to what found by Eberhard, with a numerical approach, in Ref. [13].

Case 2: left–right asymmetric measurements, $\eta_L \equiv \eta_{L,a} = \eta_{L,b}$ and $\eta_R \equiv \eta_{R,a} = \eta_{R,b}$. This is the case of atom–photon entanglement [24], for which the measurements on the atom can be done with high efficiencies. For a given efficiency η_R, Eq. (17) is satisfied when

$$\eta_L > \eta_R/[(\alpha^2+2)\eta_R - 1] \; , \tag{19}$$

and again η_L is minimized for $\alpha = 1$. Let us consider the particular case in which measurements on the right party are done with 100% efficiency. One has:

$$\eta_L > 1/(\alpha^2+1) \quad \text{when} \quad \eta_R = 1 \; . \tag{20}$$

A result analogous to our one: $\eta_L > 1/2$ when $\eta_R = 1$ and $\alpha = 1$, has been obtained in Ref. [17], but starting from a different bipartite non–maximally entangled state and by using four different measurement bases (two for each one of the two parties, as in standard Bell–tests).

Case 3: observable asymmetric measurements, $\eta_a \equiv \eta_{L,a} = \eta_{R,a}$ and $\eta_b \equiv \eta_{L,b} = \eta_{R,b}$. This is the case, for instance, of entangled neutral kaons [25], for which lifetime is measurable with a much larger efficiency than strangeness. For a given efficiency η_b, Eq. (17) is satisfied when

$$\eta_a > 2(1-\eta_b)/\alpha^2 \; . \tag{21}$$

When the measurement in the $\{|b_+\rangle, |b_-\rangle\}$ basis are possible with 100% efficiency, Eberhard inequality is violated by quantum mechanics independently of the value of η_a:

$$\text{Violation of (11)}\ \forall \eta_a \ \text{when}\ \eta_b = 1 \; . \tag{22}$$

Note that the conclusion (22) of *Case 3* is independent of the value of α, i.e., the test can be applied to an infinity of Hardy states given by Eq. (9). To have an idea of the minimum detection efficiencies required in a real test, let us first consider the case in which $\eta_b = 0.90$ and $\alpha = 0.9$ or $\alpha = \alpha_H$ (the value $\alpha_H = 0.618$ corresponds to maximize Hardy's fraction (10)): $P^{\rm I}_{\rm QM}(a_+, a_+)|_{\alpha=0.9} = 0.025$ and $\eta_a(\alpha = 0.9) > 0.25$, while $P^{\rm I}_{\rm QM}(a_+, a_+)|_{\alpha=\alpha_H} = 0.090$ and $\eta_a(\alpha = \alpha_H) > 0.52$.

In real experiments, measurements are affected by noise (i.e., by counts which do not originate from the entangled state under study) in addition to inefficiencies in the detection. For white noise, represented by a background joint probability P_B independent of the measurement settings, the state subject to observation is not Hardy's state (9) but rather the mixture:

$$\rho = (1-P_B)|\psi_H\rangle\langle\psi_H| + P_B\frac{\mathbb{1}}{4} \; . \tag{23}$$

The ratio of Eq. (17) for *Case 3* thus becomes:

$$H_{\rm QM} = \frac{(1-P_B)\dfrac{(1-\alpha^2)^2\alpha^2}{2-\alpha^2}\eta_a^2 + \dfrac{P_B}{4}}{5\dfrac{P_B}{4} + (1-P_B)\dfrac{(1-\alpha^2)^2}{2-\alpha^2}2\eta_a(1-\eta_b)} \; , \tag{24}$$

and inequality (11) is violated when, for given values of η_a and η_b, the background noise is limited by:

$$P_B \le P_B^{\max} = \frac{\dfrac{(1-\alpha^2)^2}{2-\alpha^2}\eta_a[\alpha^2\eta_a - 2(1-\eta_b)]}{1+\dfrac{(1-\alpha^2)^2}{2-\alpha^2}\eta_a[\alpha^2\eta_a - 2(1-\eta_b)]} \; . \tag{25}$$

In Figure 1 we show the maximum tolerable background noise in a loophole–free experiment adopting Eberhard inequality (11) as a function of η_a for four relevant cases. The results for $\eta_b = \eta_a$ and $\alpha = 0.99$ correspond to a critical efficiency $\eta_a^{\min} = 2/3 \simeq 0.67$, in agreement with what found in Ref. [13], while for $\eta_b = \eta_a$ and $\alpha = \alpha_H$, $\eta_a^{\min} = 4/(7-\sqrt{5}) \simeq 0.84$. Instead, when $\eta_b = 1$, and independently of the value of α, any value of η_a allows a loophole–free experiment when the background is limited to the value given by Eq. (25): for instance, for $\alpha = \alpha_H$ a genuine violation of Eberhard inequality is possible using $\eta_a > 0.33$ for a background of 1%. It turns out that for values of η_a and η_b which allow a loophole–free test, the value of α which maximizes the tolerable noise is always larger than α_H: in the limiting case of $\eta_a = \eta_b = 1$, $P_B^{\max}$ is maximum for $\alpha = \alpha_H$: $P_B^{\max}(\eta_a = \eta_b = 1, \alpha = \alpha_H) = (13-5\sqrt{5})/22 \simeq 0.083$.

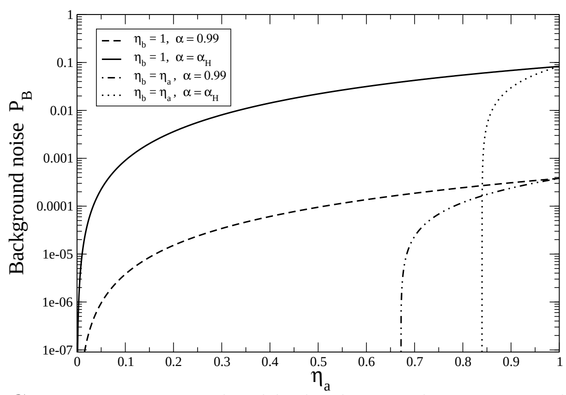

FIG. 1. Maximum tolerable background noise in a loophole–free experiment with inequality (11) as a function of η_a for *Case 3* and: $\eta_b = 1$ and $\alpha = 0.99$; $\eta_b = 1$ and $\alpha = \alpha_H = 0.618$; $\eta_b = \eta_a$ and $\alpha = 0.99$; $\eta_b = \eta_a$ and $\alpha = \alpha_H = 0.618$.

Another important question in a Bell–test is the amount of violation predicted by quantum mechanics for the Bell inequality. For our Eberhard inequality the violation is given by $V = H_{\rm QM}/H_{\rm LR}^{\max} = H_{\rm QM}$. Considering for instance atom–atom entanglement [23], for which efficiencies as high as 90% can be reached, for different values of the background noise we obtain the results of V vs α of Figure 2. For moderate noise the expected violation can be large (to be compared, for instance, with the maximum violation $(3+2\sqrt{2})/3 \simeq 1.94$ obtained for the ideal case with maximally entangled states and four measurement settings). We also note that $V \to \infty$ for $\eta_b \to 1$ and $P_B \to 0$.

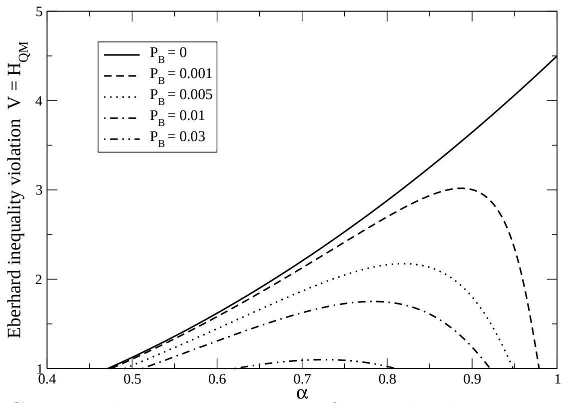

FIG. 2. Violation $V = H_{\rm QM}$ of the Eberhard inequality (11) predicted by quantum mechanics for *Case 1* with $\eta = 0.9$ as a function of the parameter α defining Hardy's state (9).

The other two Eberhard inequalities which can be derived from Hardy's reasoning applied to Eqs. (3), (4), (5) and (10) differ from the one in Eq. (11) for the two joint probabilities involving undetected events: one contains the sum $P(a_+, b_0) + P(b_0, b_+)$, the other $P(b_+, b_0)+P(b_0, a_+)$. We do not discuss these additional inequalities since, for each one of the three special cases previously analyzed, a loophole–free test with each one of them would require values for the detection efficiency thresholds and the tolerable noise which are less convenient than obtained for inequality (11).

In conclusion, we have discussed a Bell–type test involving a bipartite non–maximally entangled state of the Hardy type and (unlike standard Bell–tests) the same pair of measurement bases for both parties. As far as we know, the results we have obtained improve all previous discussions aimed at finding the bipartite entangled state and the Bell measurements bases allowing one to refute local realism with the minimum possible detection efficiencies. In the design of new detection–loophole–free tests it is important to identify entangled systems for which one of the two required observables can be measured with very high efficiency: in the absence of noise, a genuine Bell inequality violation is thus affordable even with very low efficiencies for the other measurement.

The only system we know that enables observable asymmetric measurements consists of entangled neutral kaon pairs [25]. For kaons, lifetime measurements can be performed quite efficiently ($\eta_b \simeq 0.9$), but, unfortunately, strangeness measurements are still affected by very small efficiencies ($\eta_a < 0.01$). On the contrary, for atom–atom entanglement the proposed Bell–test allows large falsifications of local realism (even greater than the well–known violations predicted for maximally entangled states in ideal Bell–tests).

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